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Standing localized modes in nonlinear lattices

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The theory of standing localized modes in discrete nonlinear lattices is presented. We start from a rather general model describing a chain of particles subjected to an external (on-site) potential with cubic and quartic nonlinearities (the so-called discrete Klein-Gordon model), and, using the approximation based on *the discrete nonlinear Schrödinger equation*, derive a system of two coupled nonlinear equations for slowly varying envelopes of two counterpropagating waves of the same frequency. We show that spatially localized modes exist in the frequency–wave number domain where the lattice displays modulational instability; two families of localized modes are found for this case as separatrix solutions of the effective equations for the wave envelopes. When the nonlinear plane wave in the lattice is stable to small modulations of its amplitude, nonlinear localized modes appear as dark solitons associated with the so-called extended modulational instability. These localized modes may be treated as domain walls or kinks connecting two standing plane-wave modes of the similar structure. We investigate analytically and numerically the special family of such localized solutions that, in the vicinity of the zero-dispersion point, cover exactly the case of the so-called *self-induced gap solitons* recently introduced by Kivshar [Phys. Rev. Lett. **70**, 3055 (1993)]. Application of the theory to the case of parametrically driven damped lattices is also briefly discussed, and it is mentioned that some of the solutions considered in the present paper may be extended to include the case of localized modes in driven damped lattices, provided the mode frequency and amplitude are fixed by the parameters of the external parametric ac force.

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I. INTRODUCTION

As is well known, models describing microscopic phenomena in solids are inherently *discrete*, and discreteness effects may drastically modify the nonlinear dynamics and properties of spatially localized modes (see, e.g., some examples in Refs. [1–8]). For excitations that vary slowly on the scale of the lattice spacing, one can approximate discrete models by continuum partial differential equations, obtaining analytical solutions which are close to the phenomena observed in original discrete (but often analytically intractable) models. The well-known example is lattice solitons which in some particular cases are described by the Korteweg-de Vries equation for relative particle displacements or the nonlinear Schrödinger (NLS) equation for the carrier wave envelope (see, e.g., Ref. [3]). In the former case, a more rigorous approach (which takes into account effects of discreteness) allows the determination of the soliton velocity with a higher accuracy than from the continuous approximation, but it does not greatly alter the conditions for the soliton existence and propagation [9,10]. In the latter case, discreteness of the primary chain may be partially tractable through the discrete carrier wave (when taken as an exact traveling-wave solution of the lattice equations), assuming that due to nonlinearity the wave envelope is slowly changing in time and space. As a result, it may be shown that the slow variations of the envelope are described by an effective NLS equation (see, e.g., Ref. [11] and references therein). It is natural to name this type of envelope solitons *solitons on a traveling carrier wave.* On the other hand, linear theory of wave propagation considers also the so-called *standing modes* which appear as a result of linear superposition of two counterpropagating waves, i.e., two waves with the same frequency but opposite wave numbers. As a matter of fact, properties of linear standing waves are *exactly the same* as those of linear traveling waves, and usually they do not attract too much attention in the theory of wave propagation in linear lattices. However, in the case of nonlinear lattices these two types of waves, traveling and standing waves, may drastically differ from each other. The interest in the standing localized modes in nonlinear lattices has been initiated particularly by the recent experimental observation of standing localized modes in a chain of parametrically driven pendulums [12]. Indeed, applying an external parametric force to a nonlinear chain, one may excite *two* waves of the same frequency ω (equal, e.g., to a half of the frequency of the external force in the case of the parametrically driven chain) but with opposite wave numbers, $\pm k_0$, because in the typical case of systems with a symmetric spectrum band the standard property $\omega(-k_0) = \omega(k_0)$ means that at least two wave numbers correspond to a single value of the wave frequency ω. As a result, standing nonlinear modes are excited in a natural way and may be also supported by

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the external driving force in a lossy system. As follows from the present analysis, properties of standing and traveling waves in nonlinear lattices differ drastically because *standing modes are created by two counterpropagating waves* which, as a matter of fact, *strongly interact through mutual nonlinearity-induced coupling.*

Up to present time, standing localized modes in discrete lattices have been not investigated analytically in detail. We would like to mention, however, that the nonlinear theory of standing localized modes in the continuous approximation has been recently considered in Refs. [12,13], the simplest bright and dark solitons for discrete lattices have been presented in Ref. [14], and the theory of standing waves corresponding to the so-called wavelength-four modes has been proposed in Refs. [15,16]. The purpose of this paper is to present the theory of standing localized modes in nonlinear lattices which, first, takes into account the effects produced by lattice discreteness of the model and, second, covers all the particular cases analyzed up to now. One of the main features observed in nonlinear lattices, i.e., the dependence of modulational instability on the wave number of the carrier wave, should also be taken into account by such a theory because this effect certainly leads to a change of properties of localized modes in the lattice. For example, depending on the wave number of the carrier wave nonlinear modes may be spatially localized or they may exist as kink-profile structures on a modulationally stable (standing) background wave. To describe the main properties of nonlinear standing modes in discrete lattices, in the present paper we consider, as a typical example, a simple model describing a chain of particles subjected to cubic and quartic nonlinear (on-site) potential and, using an approximation based on the discrete NLS equation, we show that the standing nonlinear modes are described by a system of two coupled equations. We present different families of localized solutions to these equations including the case of the so-called self-induced gap solitons previously analyzed in Refs. [15,16]. We believe the results obtained will allow a deeper insight to be gained into the properties of a variety of nonlinear localized modes observed experimentally in a parametrically driven chain of pendulums [12] as well as to demonstrate the general features and properties of standing localized modes in discrete lattices.

The paper is organized as follows. In Sec. II we present our model which may be reduced to the discrete NLS equation in the case when the interparticle coupling in the chain is weak. We also briefly discuss modulational instability for discrete lattices and the well-known case of envelope (bright and dark) *solitons on a traveling carrier wave* to make a subsequent comparison with the theory of standing localized modes. In Sec. III we derive the system of two coupled NLS type equations that describes properties of standing localized modes in the case when the envelopes of two counterpropagating waves, which form a standing mode, are slowly varying on the scale of the inverse wave number of the carrier wave. As we show in Sec. IV, the equations proposed in the paper also cover the recently analyzed case of the wavelength-four modes. In fact, by a simple transformation we reduce the two coupled NLS equations into the system of equations derived earlier by Kivshar [15,16] using a completely different approach. The case when the carrier wave number is far from the zero-dispersion point and from the edges of the Brillouin zone is analyzed in Sec. V where we demonstrate that the resulting system of the coupled NLS equations displays a family of more general localized solutions which are found numerically. Some extensions of the theory to cover the case of parametrically driven damped nonlinear chains are briefly discussed in Sec. VI. At last, Sec. VII concludes the paper with a general discussion of the results and a brief summary of the open problems.

II. MODEL

We consider the dynamics of a one-dimensional chain of atoms with the mass m, harmonically coupled to their neighbors, and subjected to a cubic and quartic external nonlinear (on-site) potential. Denoting by $u_n(t)$ the displacement of atom n, its equation of motion is written in the form

$$m\frac{d^2u_n}{dt^2} - k_2(u_{n+1} + u_{n-1} - 2u_n) + m\omega_0^2 u_n + \alpha u_n^2 + \beta u_n^3 = 0, \quad (1)$$

where k_2 is the coupling constant characterizing a strength of the interparticle forces, ω_0 is the frequency of small-amplitude vibrations in a well of the substrate potential, α and β are the anharmonicity parameters of the on-site potential. Linear waves in the lattice are characterized by the frequency spectrum $\omega(q)$ which has a gap ω_0 and is limited by the cutoff frequency $\omega_{\max} = (\omega_0^2 + 4k_2/m)^{1/2}$ due to discreteness.

Analyzing slow temporal variations of the wave envelope, we try to retain in full the discreteness of the primary model. This is possible indeed under the condition $m\omega_0^2 \gg 4k_2$, i.e., when a (linear) coupling force between the particles is weak. Looking for the solution in the form

$$u_n = \phi_n + \psi_n e^{-i\omega_0 t} + \xi_n e^{-2i\omega_0 t} + \cdots + \text{c.c.}, \quad (2)$$

where c.c. stands for the complex conjugate terms, and keeping only the lowest order terms in rapidly varying oscillations (see details of the corresponding justification in Ref. [17]), we obtain the nonlinear equation for ψ_n,

$$2i\omega_0 m\frac{d\psi_n}{dt} + k_2(\psi_{n+1} + \psi_{n-1} - 2\psi_n) - 2\alpha - (\phi_n\psi_n + \psi_n^*\xi_n) - 3\beta|\psi_n|^2\psi_n = 0, \quad (3)$$

and two algebraic relations for ϕ_n and ξ_n,

$$\phi_n \approx -\frac{2\alpha}{\omega_0^2}|\psi_n|^2, \quad \xi_n \approx \frac{\alpha}{3\omega_0^2}\psi_n^2. \quad (4)$$

The results (3) and (4) are generalizations of the well-known results for the continuum case (see, e.g., Ref. [11]). Thus, the final discrete NLS equation stands

$$i\frac{d\psi_n}{dt} + K(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + \lambda|\psi_n|^2\psi_n = 0, \quad (5)$$

where

$$K = k_2/2m\omega_0, \quad \lambda = \frac{1}{2m\omega_0}\left(\frac{10\alpha^2}{3\omega_0^2} - 3\beta\right). \quad (6)$$

Equation (5) is used below to analyze different types of localized modes in the chain.

As is well known, nonlinear systems may exhibit an instability that leads to a growth of self-induced modulations of the continuous-wave (cw) mode as a result of an interplay between nonlinear and dispersive effects, and it is also responsible for energy localization and the formation of localized pulses. For the discrete NLS equation (5), derived in the single-frequency approximation, modulational instability was analyzed by Kivshar and Peyrard [7]. Contrary to what would be found in the continuum limit, the stability criterion depends on the carrier wave number q, and this property is very common for different discrete models (see, e.g., Ref. [18]). The region of instability appears only if

$$\lambda \cos(qa) > 0. \quad (7)$$

For positive λ and a given q, a plane wave will be *unstable* in the region $q < \pi/2a$ and stable otherwise [7].

The standard case which has been analyzed in the literature for various nonlinear models is the so-called solitons on a traveling carrier wave that arises when a carrier wave supporting solitons is treated in the so-called *discrete-carrier-wave approximation.* To make such an analysis for the model (5), let us derive the effective nonlinear equation which describes nonlinearity-induced modulations of the wave envelope which is, according to the results of the previous section, modulationally unstable provided the condition (7) holds. We are looking for a solution of Eq. (5) in the form

$$\psi_n(t) = \Psi(n,t)\exp(iqna - i\omega t), \quad (8)$$

where for small nonlinearities it is assumed that the wave number q and frequency ω are connected by the *linear* dispersion relation $\omega = 4K\sin^2(qa/2)$. Substituting Eq. (8) into Eq. (5) and keeping only the first three terms of the Taylor expansion of the functions $\Psi(n \pm 1, t)$ (which are assumed to be slowly varying), we come to the equation

$$i\frac{\partial\Psi}{\partial t} + iV_g\frac{\partial\Psi}{\partial x} + \Delta\frac{\partial^2\Psi}{\partial x^2} + \lambda|\Psi|^2\Psi = 0, \quad (9)$$

where

$$V_g \equiv \frac{d\omega}{dq} = 2aK\sin(qa) \quad (10)$$

is the group velocity of linear waves in the lattice NLS equation (5), and the parameter

$$\Delta \equiv \frac{d^2\omega}{dq^2} = a^2K\cos(qa) \quad (11)$$

describes the group-velocity dispersion of linear waves. Equation (9) is the standard continuous NLS equation which describes modulations of the wave envelope in the reference frame moving at the group velocity V_g. We refer to the solitons described by this equation as *solitons on a traveling carrier wave.* We would like to mention again that such an approach is commonly used to analyze nonlinear waves in different lattice models of solids (see, e.g., Refs. [3,11] to cite a few). In that case the properties of the carrier wave already include discreteness of the primary model, because the wave number of the carrier wave is assumed to be taken within the whole region of the Brillouin zone, $0 < q < \pi/a$.

The NLS equation (9) is exactly integrable and its localized soliton solutions are well known: they are *bright solitons* for $\lambda\Delta > 0$ and *dark solitons*, i.e., solitons on a modulationally stable background, for $\lambda\Delta < 0$. The main property of such localized solutions is the following: They exist as localized structures of *a traveling carrier wave*, and therefore they are characterized by the group velocity V_g. We would like to note, however, that this approach allows also the description of the two simplest classes of standing localized structures when the wave number q approaches the edges of the Brillouin zone (0 or π/a), where the group velocity V_g vanishes. In the former case, i.e., $q = 0$, the envelope $\Psi(n,t)$ describes slow variations of the field ψ_n itself, and in the latter case, the function $\Psi(n,t)$ describes an envelope of the out-of-phase vibrations in the lattice, $\psi_n(t) = (-1)^n\Psi(n,t)$, where $\Psi(n,t)$ is a solution of Eq. (9) at $V_g = 0$.

III. COUPLED-MODE EQUATIONS

To describe nonlinear modulations of the standing carrier wave, we look for solutions of Eq. (5) in the form,

$$\psi_n = \Psi_1(n,t)e^{i\theta_+} + \Psi_2(n,t)e^{i\theta_-}, \quad (12)$$

where

$$\theta_\pm = -i\omega t \pm iqna, \quad (13)$$

and assume that the envelope functions Ψ_1 and Ψ_2 of two counterpropagating waves vary slowly in space and time.

The physical motivation for analyzing the interaction of two counterpropagating waves of the same frequency but opposite wave numbers is the following: let us consider an external (parametric or direct) driving force of the frequency ω_e which is selected within the spectrum band of the lattice. Such a force, being applied to the lattice, excites lattice oscillations at the frequency ω_e, for a direct force, or at $\omega_e/2$, for a parametric force. If the linear spectrum band $\omega(k)$ is symmetric, i.e., $\omega(-k) = \omega(k)$, this force generates *two waves* with the wave numbers k_0

and $-k_0$ (see Fig. 1), which may form a standing localized mode through nonlinear interaction of two traveling modes propagating with two opposite group velocities $\pm V_g(k_0)$.

Substituting Eqs. (12) and (13) into Eq. (5) and combining the terms proportional to $\exp(i\theta_+)$ or $\exp(i\theta_-)$, we obtain the system of two coupled NLS equations,

$$i\frac{\partial}{\partial t}\Psi_1 + iV_g\frac{\partial}{\partial x}\Psi_1 + \Delta\frac{\partial^2}{\partial x^2}\Psi_1 + \lambda(|\Psi_1|^2\Psi_1 + 2|\Psi_2|^2\Psi_1 + \Psi_2^2\Psi_1^* e^{-4iqna}) = 0, \quad (14)$$

$$i\frac{\partial}{\partial t}\Psi_2 - iV_g\frac{\partial}{\partial x}\Psi_2 + \Delta\frac{\partial^2}{\partial x^2}\Psi_2 + \lambda(|\Psi_2|^2\Psi_2 + 2|\Psi_1|^2\Psi_2 + \Psi_1^2\Psi_2^* e^{4iqna}) = 0, \quad (15)$$

where V_g and Δ are defined in Eqs. (10) and (11) and in the derivatives written above the variable $x \approx na$ is treated as continuous. We keep, however, the explicit phase in the last terms in Eqs. (14) and (15) to make the subsequent analysis more clear. The accuracy of the derivation of Eqs. (14) and (15) is exactly the same as for Eq. (9), in the latter case the full form of the asymptotic expansion has been discussed earlier (see, e.g., Ref. [11]). However, we would like to note that we did not make any additional assumption to derive the system of the coupled NLS equations (14), (15) from the discrete NLS equation (5) except that of slowly varying envelopes of two counterpropagating waves, this allows the omission of all higher-order derivatives in the corresponding Taylor series.

The system similar to that given by Eqs. (14) and (15) is known in nonlinear optics, e.g., in the theory of the so-called nonlinear birefringent fibers, and it describes the coherent and incoherent interaction of two optical polarizations in Kerr media when the phase and group velocities of two polarization components differ due to the effect of birefringence [19]. As has been mentioned in Ref. [19], for the case of optical solitons in birefringent fibers, the terms which have oscillating multipliers

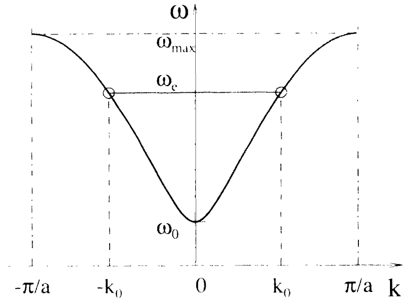

FIG. 1. Eigenfrequency spectrum of linear waves in the chain (1). If an external force with the frequency ω_e is applied to the lattice, it excites two traveling waves with the wave numbers $\pm k_0$ which may form a standing nonlinear mode.

$\sim \exp(\pm 4iqx)$ may be neglected because, according to Ref. [19], for realistic physical parameters they are always rapidly oscillating. In this latter case, the system (14), (15) possesses vector soliton solutions. For example, for $\Delta > 0$ the soliton solutions were found to be of the form [19]

$$\Psi_{1,2}(x,t) = \frac{A\, e^{i\Delta B^2 t}}{\cosh(Bx)} \exp\left(\frac{iV_g^2}{4\Delta}t \mp \frac{iV_g}{2\Delta}x\right), \quad (16)$$

where $B^2 = (3\lambda/2\Delta)A^2$ and A is the soliton amplitude. The two solitons (16) have a phase shift which is induced by the nonzero group velocity, and this shift compensates exactly the so-called "walk-off" effect between the two polarization modes. As was shown numerically [19] and analytically [20], nonlinearity may stabilize the partial solitons against both spreading due to dispersion and splitting due to difference in their group velocities. Above a certain threshold amplitude, the fractional pulses in each of the two polarization modes trap each other and move together as one vector object.

In the case of nonlinear lattices, the wave number q is selected within the Brillouin zone ($0 < q < \pi/a$), so that the assumption that the terms containing oscillating multipliers are small is not valid in the vicinity of the zero-dispersion point $q = \pi/2a$. Thus, we should analyze Eqs. (14) and (15) in a more general form. As is shown below, this is very important in describing the so-called self-induced gap solitons corresponding to the wavelength-four carrier wave.

IV. WAVELENGTH-FOUR MODES

In the case when the carrier wave is selected as a wavelength-four mode of the linear spectrum, $q = \pi/2a$, we have $e^{2i\pi n} \equiv 1$, so that all nonlinear terms in Eqs. (14) and (15) are important. In this case, the system of Eqs. (14) and (15) takes the form

$$i\frac{\partial}{\partial t}\Psi_1 + iV_0\frac{\partial}{\partial x}\Psi_1 + \lambda(|\Psi_1|^2\Psi_1 + 2|\Psi_2|^2\Psi_1 + \Psi_2^2\Psi_1^*) = 0, \quad (17)$$

$$i\frac{\partial}{\partial t}\Psi_2 - iV_0\frac{\partial}{\partial x}\Psi_2 + \lambda(|\Psi_2|^2\Psi_2 + 2|\Psi_1|^2\Psi_2 + \Psi_1^2\Psi_2^*) = 0, \quad (18)$$

where $\Delta(\pi/2) = 0$ and $V_0 \equiv V_g(\pi/2)$ are the values of the second-order dispersion and group velocity calculated at $q = \pi/2a$, and Δ vanishes at that point. Let us now use the transformations

$$\Psi_{1,2} = \frac{1}{2}(v \pm iw). \quad (19)$$

As a matter of fact, the new functions v and w introduced above describe slowly varying envelopes of two different groups of particles in a lattice, odd and even particles [15,16]. It is easy to verify that the combinations of the nonlinear terms in Eqs. (17) and (18) are transformed to be

$$|\Psi_1|^2\Psi_1 + 2|\Psi_2|^2\Psi_1 + \Psi_2^2\Psi_1^* = |v|^2 v + i|w|^2 w, \quad (20)$$

$$|\Psi_2|^2\Psi_2 + 2|\Psi_1|^2\Psi_2 + \Psi_1^2\Psi_2^* = |v|^2 v - i|w|^2 w. \quad (21)$$

Therefore, the system of the coupled NLS equations (17), (18) for the functions w and v takes the form

$$i\frac{\partial v}{\partial t} - V_0\frac{\partial w}{\partial x} + \lambda|v|^2 v = 0, \quad (22)$$

$$i\frac{\partial w}{\partial t} + V_0\frac{\partial v}{\partial x} + \lambda|w|^2 w = 0. \quad (23)$$

Looking for the spectrum of the cw solutions to this nonlinear system, we find the result

$$(\tilde{\omega} - \lambda v_0^2)(\tilde{\omega} - \lambda w_0^2) = V_0^2\tilde{q}^2, \quad (24)$$

where $\tilde{\omega}$ and $\tilde{q}$ are the frequency and wave number of the cw solutions of two modes with the amplitudes w_0 and v_0, respectively. The dispersion relation (24) exhibits *a nonlinearity-induced gap* in the cw spectrum and this gap is proportional to the difference in the amplitudes of odd and even particle oscillations,

$$\delta\tilde{\omega} = \lambda|v_0^2 - w_0^2|. \quad (25)$$

As has been discussed in Refs. [15,16], the gap in the nonlinear spectrum may be a factor of the wave localization at $q = \pi/2a$ provided the nonlinearity is large enough. However, this kind of localized structure has to differ drastically from the standard spatially localized modes in nonlinear models. Indeed, both the wave field components cannot vanish in the same direction because there is no gap in the linear spectrum and small-amplitude oscillations at that frequency will be delocalized.

Analyzing this kind of localized structures, we follow Refs. [15,16] and look for stationary solutions of Eqs. (22) and (23) in the form

$$(v, w) \propto (f_1, f_2)e^{i\kappa t}. \quad (26)$$

The stationary solutions are described by the system of two ordinary differential equations of the first order,

$$\frac{df_1}{dz} = \kappa f_2 - \lambda f_2^3, \quad (27)$$

$$\frac{df_2}{dz} = -\kappa f_1 + \lambda f_1^3, \quad (28)$$

where $z \equiv x/V_0$. Equations (27), (28) describe the dynamics of a Hamiltonian system with one degree of freedom and the conserved energy,

$$E = \frac{1}{2}\kappa(f_1^2 + f_2^2) - \frac{1}{4}\lambda(f_1^4 + f_2^4), \quad (29)$$

and they may be integrated with the help of the auxiliary function $g = (f_1/f_2)$, for which the following equation is valid,

$$\left(\frac{dg}{dz}\right)^2 = \kappa^2(1+g^2)^2 - 4\lambda E(1+g^4). \quad (30)$$

Different kinds of solutions of Eq. (30) are characterized by different values of the energy E. On the phase plane (f_1, f_2) (see Fig. 2) soliton solutions correspond to the separatrix curves connecting a pair of the neighboring saddle points $(0, f_0)$, $(0, -f_0)$, $(f_0, 0)$, or $(-f_0, 0)$, where $f_0^2 = \kappa/\lambda$. Calculating the value of E for these separatrix solutions, $E = \kappa^2/4\lambda$, it is possible to integrate Eq. (30) in elementary functions and to find the explicit form of localized solutions,

$$g(z) = \exp(\pm\sqrt{2}\kappa z), \quad (31)$$

$$f_2^2 = \frac{\kappa e^{\mp\sqrt{2}\kappa z}[2\cosh(\sqrt{2}\kappa z) \pm \sqrt{2}]}{2\lambda\cosh(2\sqrt{2}\kappa z)}, \qquad f_1 = g f_2. \quad (32)$$

The solutions (31) and (32), but for negative κ, exist also for defocusing nonlinearity when $\lambda < 0$.

The results (31), (32) give the shapes of the localized modes in the discrete nonlinear lattice. Because all combinations of the signs are possible in Eq. (32), there are *four* solutions of this type and two of them are presented in Figs. 3(a) and 3(b) (the other two modes are obtained by a trivial change of the sign). The resulting localized structure represents two kinks in the two oscillating lattice modes which are composed of the opposite [see Fig. 3(a)] or the same [see Fig. 3(b)] polarities, so that both of them cannot be localized in one direction. This is a direct consequence of the nonlinearity-induced gap (25) in the cw spectrum (24), and this gap disappears in the linear limit (see more detailed discussion of the nonlinearity-induced gap in Ref. [16]).

The existence of the nonlinearity-induced gap explains why at the point $q = \pi/2a$, where the second-order dispersion term exactly vanishes, we may neglect the third-

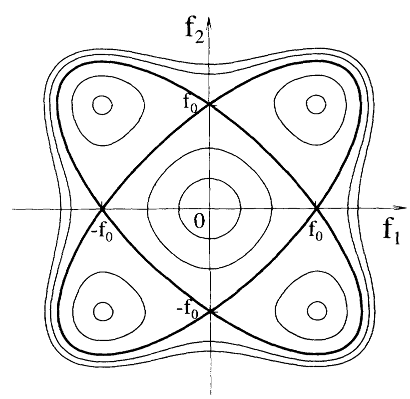

FIG. 2. Phase plane of the system (27) and (28). The separatrix curves connect pairs of four saddle points $(0, f_0)$, $(f_0, 0)$, $(0, -f_0)$, and $(-f_0, 0)$.

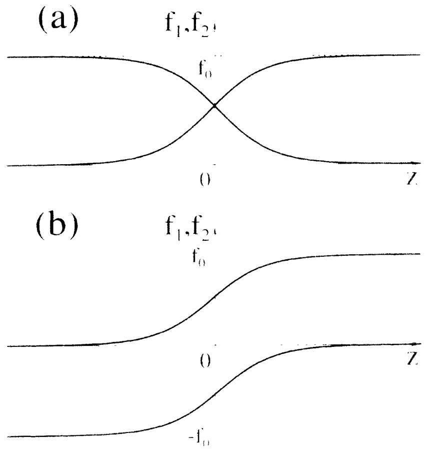

FIG. 3. Kink-type localized modes corresponding to the separatrix curves shown in Fig. 2. The solutions are given by Eqs. (31) and (32).

order terms $\partial^3\Psi_1/\partial x^3$ and $\partial^3\Psi_2/\partial x^3$ in Eqs. (17) and (18), respectively. Indeed, for the linearized version of Eqs. (17), (18) the linear spectrum has no dispersion but nonlinearity can produce an effective dispersion opening the nonlinear gap in the spectrum of the cw solutions as is shown in Eq. (25). This means that in the nonlinear case when two components are nonlinearly coupled, the first-order derivative terms in Eqs. (17), (18) will define the parameters of localized solutions and a contribution of the third-order terms will be negligible, provided the continuum approximation is valid, i.e., for $a^2 \ll L^2$, where L is the characteristic scale of the localized mode (31), (32) which is $L = V_0/\kappa$. This case is different from the one-component NLS equation (9) briefly discussed in Sec. II. At the point $q = \pi/2a$, where the second-order derivative term vanishes, the first-order derivative cannot compensate for nonlinearity (this is simply the group-velocity term) and localized solutions may appear only due to the effect of the third-order dispersion which itself defines a spatial extension of localized modes (see, e.g., as an example, the case of the zero-dispersion point in the context of nonlinear optical fibers [21]). From the viewpoint of the nonlinear gap discussed above, for a traveling wave of a single NLS equation the only effect of nonlinearity is to shift the wave frequency but do not open a gap, so that in this case the effective contribution of dispersion must be taken as a higher-order term.

V. GENERAL CASE

A. Basic equations

Let us now describe a more general situation ($q \neq \pi/2a$) when the frequency of the carrier wave does not correspond to the zero-dispersion point. This time the terms in Eqs. (14) and (15) containing multipliers $\exp(\pm 4iqna)$ are rapidly oscillating, and they may now be omitted. Making the transformation

$$\Psi_{1,2} = \frac{1}{\sqrt{\lambda}}\Phi_{1,2}\exp\left(-\frac{iV_g^2}{4|\Delta|}t \pm \frac{iV_g}{2|\Delta|}x\right) \qquad (33)$$

and renormalizing the variable, $x \to x\sqrt{|\Delta|}$, we reduce the system (14), (15) to the following one,

$$i\frac{\partial\Phi_1}{\partial t} + \sigma\frac{\partial^2\Phi_1}{\partial x^2} + (|\Phi_1|^2 + 2|\Phi_2|^2)\Phi_1 = 0, \qquad (34)$$

$$i\frac{\partial\Phi_2}{\partial t} + \sigma\frac{\partial^2\Phi_2}{\partial x^2} + (|\Phi_2|^2 + 2|\Phi_1|^2)\Phi_2 = 0, \qquad (35)$$

where $\sigma \equiv \mathrm{sign}(\lambda\Delta)$. We postulate the existence of spatially localized modes in which both counterpropagating components have the same frequency. We then look for stationary solutions in the form

$$\Phi_1 = \phi_1(x)e^{i\Omega t}, \quad \Phi_2 = \varphi_2(x)e^{i\Omega t}. \qquad (36)$$

where the envelopes ϕ_1 and ϕ_2 are assumed to be real functions. Substituting Eq. (36) into Eqs. (34) and (35) leads to the system of two coupled ordinary differential equations,

$$\sigma\frac{d^2\phi_1}{dx^2} = \Omega\phi_1 - \phi_1^3 - 2\phi_2^2\phi_1, \qquad (37)$$

$$\sigma\frac{d^2\phi_2}{dx^2} = \Omega\phi_2 - \phi_2^3 - 2\phi_1^2\phi_2. \qquad (38)$$

In a mechanical analog picture Eqs. (37) and (38) are the motion equations of a unit-mass particle on the plane (ϕ_1, ϕ_2) in the effective potential

$$V(\phi_1,\phi_2) = \sigma\left[-\frac{1}{2}\Omega(\phi_1^2+\phi_2^2) + \frac{1}{4}(\phi_1^4+\phi_2^4) + \phi_1^2\phi_2^2\right]. \qquad (39)$$

The bound solitary-wave solutions are then given by the separatrix trajectories on this potential [22]. The dispersion coefficient σ determines the properties of the potential extrema, i.e., maxima, minima, and saddle points. Since the separatrices are the trajectories which connect such points, we have to consider the cases of positive ($\sigma = +1$) and negative ($\sigma = -1$) dispersion separately.

B. Positive dispersion

In the case where $\sigma = +1$, the potential $V(\phi_1, \varphi_2)$ possesses a unique maximum at the origin surrounded by a valley in which there are four minima of the axes, $\phi_1 = 0$ and $\phi_2 = 0$. The separatrices are then the trajectories which start and end at the origin.

Being symmetric with respect to the axes $\phi_1 = 0$, $\phi_2 = 0$, the potential possesses simple straight line separatrices along these axes. Substituting $\phi_2 = 0$ (or $\phi_1 = 0$) in Eq. (37) [or Eq. (38)], we find the corresponding simplest separatrix solutions which correspond to soliton-wave envelopes,

$$\phi_1(x) = \frac{\sqrt{2\Omega}}{\cosh(\sqrt{\Omega}x)}, \quad \phi_2(x) = 0, \qquad (40)$$

or

$$\phi_1(x) = 0, \quad \phi_2(x) = \frac{\sqrt{2\Omega}}{\cosh(\sqrt{\Omega}x)}. \tag{41}$$

These solutions represent the case when the amplitude of one of the counterpropagating modes vanishes, and the soliton solution is an envelope of a traveling carrier wave only, as it was discussed in Sec. II.C.

Analyzing solutions of a more general type, we note that between the minima, the potential $V(\phi_1, \phi_2)$ exhibits four saddle points on the bisecting lines $\phi_1 = \pm\phi_2$. Being axially symmetric with respect to these lines, the potential also exhibits separatrices along them. Setting $\phi_1 = \pm\phi_2$ in Eqs. (37) and (38), we easily find the corresponding solitary-wave solutions,

$$\phi_1 = \pm\phi_2 = \frac{\sqrt{2\Omega/3}}{\cosh(\sqrt{\Omega}x)}. \tag{42}$$

These solutions are nothing but the composite (standing carrier wave) solitons mentioned in Sec. III and expressed here in dimensionless units. Such solitons describe mutual trapping of two envelope solitons belonging to different counterpropagating modes of lattice vibrations.

By means of the standard shooting technique applied to Eqs. (37) and (38), we have investigated numerically the existence of other separatrix curves on the potential $V(\phi_1, \phi_2)$. Two kinds of numerically found trajectories were identified: The trajectories which form closed loops being axially symmetric with respect to the bisecting lines $\phi_1 = \pm\phi_2$, and the trajectories which fold back upon themselves showing no particular symmetry.

Two examples of such separatrices belonging to both these families are shown in Figs. 4(a) and 4(b) together with the corresponding solitary-wave envelopes. The first are characterized by having an odd number of zeroes of the envelopes of both components whereas the envelopes of the second family exhibit even numbers of zeroes. These results show that the mutual nonlinear trapping of counterpropagating waves in nonlinear lattices may lead to complex localized structures which have no simple analog in the theory of traveling waves. Of course, different solutions of these two families have different energies, so that for the driven damped lattices it is probable that only the lowest-order mode may be observable. These questions will be addressed in our future work.

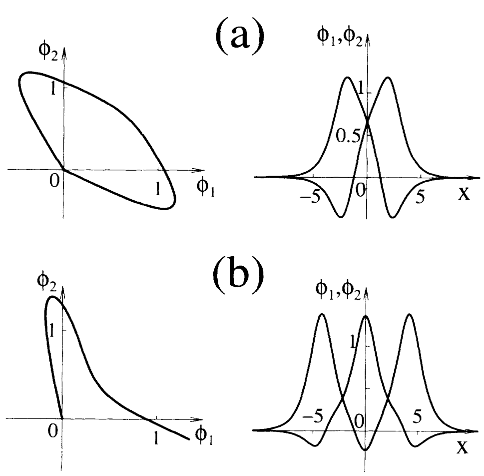

FIG. 4. Examples of the separatrix curves for the potential $V(\phi_1, \phi_2)$ (left column) and the corresponding solitary wave envelopes (right column). The case (a) presents the family of the solutions corresponding to symmetric separatrices, and the case (b)—solutions with no special symmetry of the corresponding separatrices.

C. Negative dispersion

1. *Extended modulational instability*

The case of the negative dispersion is of particular interest here. In the presence of only one carrier wave, i.e., $\phi_1 = 0$ or $\phi_2 = 0$, Eqs. (34) and (35) reduce to the single NLS equation which is modulationally stable provided $\sigma = -1$. However, when two counterpropagating waves are present, there is an incoherent coupling between them which is characterized by a cross-phase modulation twice as large as the self-phase modulation. Since the early work of Berkhoer and Zakharov [23], modulational instability in systems with incoherently coupled NLS equations has attracted much attention. In the context of the results of Ref. [23], incoherent coupling represents the process of cross-phase modulation between the two polarization components of a transverse electromagnetic wave propagating in a Kerr-type medium. Berkhoer and Zakharov showed, in particular, that cross-phase modulation is responsible for the onset of modulational instability in the negative dispersion regime. In other words, they showed that incoherent coupling between two NLS equations leads to *an extension of modulational instability* to the parameter domains in which both waves in separation are modulationally stable. Incoherent coupling and extended modulational instability occur in a variety of different physical contexts and, in particular, in plasma physics and nonlinear optics, for example, in the propagation of transverse electromagnetic waves in cold plasmas [23,24], in the interaction between Langmuir and ion-acoustic waves [25], in the propagation of light in magneto-optically active media [26] or nonlinear dielectrics [27].

The phenomenon of extended modulational instability has been recently reconsidered by Haelterman and Sheppard [28]. They showed, in particular, that extended modulational instability is associated with the existence of solitary waves of dark-profile type in the same way as modulational instability in the single NLS equation is associated with the bright envelope solitons (see, e.g., Ref. [29]). The link between the extended modulational instability and this type of solitary wave is established as follows: The dynamics of the extended modulational instability and the corresponding family of space- and time-periodic solutions are studied using the approximate three-wave model for the coupled NLS equations. This analysis reveals the existence of stationary periodic solutions analogous to the cnoidal wave solutions of the single NLS equation. By means of numerical study of the coupled NLS equations it was shown that these particular solutions tend to a solitary wave as their period is increased to infinity, exactly as the cnoidal waves tend to the bright soliton in the case of the single NLS equation [29]. On the basis of this reasoning the dark solitary wave can be viewed as the solitons associated with the extended modulational instability [28].

This theory applies here to the counterpropagating waves in nonlinear lattices. As a matter of fact, two such waves are modulationally unstable in a lattice when they propagate together and they can form a stable kink type standing wave, or dark-profile soliton, which involves both counterpropagating modes.

2. Composite kink modes

In the negative dispersion regime ($\sigma = -1$) the potential $V(\phi_1, \phi_2)$ possesses four maxima on the axes $\phi_1 = 0$ and $\phi_2 = 0$, and the origin is now a minimum. The separatrices are then trajectories which connect pairs of maxima. Due to the axial symmetry of the potential, the separatrices connecting pairs of opposite maxima are straight lines along the axes (see Fig. 5, dashed lines) and they may be easily found analytically. Setting $\phi_2 = 0$ (or $\phi_1 = 0$) in Eqs. (37) and (38), we find

$$\phi_1 = \sqrt{\Omega}\tanh(\sqrt{\Omega/2}x), \quad \phi_2 = 0, \tag{43}$$

or

$$\phi_1 = 0, \quad \phi_2 = \sqrt{\Omega}\tanh(\sqrt{\Omega/2}x). \tag{44}$$

These are the usual fundamental dark solitons to the single NLS equation described in Sec. II C.

Obviously, as in the case of positive dispersion the potential is symmetric with respect to the bisecting lines $\phi_1 = \pm\phi_2$ on which it possesses saddle points (see Fig. 5, dotted lines). The trajectories connecting these saddle points are also separatrices of the potential V. The corresponding solitary waves are easily calculated from Eqs. (37) and (38) when setting $\phi_1 = \pm\phi_2$. We obtain

$$\phi_1 = \pm\phi_2 = \sqrt{\Omega/3}\tanh(\sqrt{\Omega/2}x). \tag{45}$$

These composite kink solitons represent mutually trapped dark solitons belonging to different counterpropagating modes of lattice vibrations. This situation is analogous to that obtained with bright solitons in the positive dispersion regime.

Of special interest here are the separatrices which connect adjacent maxima (see Fig. 5, solid lines). Such separatrices are characteristic of the topology of the potential V with $\sigma = -1$ and they have no counterpart in the case of positive dispersion. Because Eqs. (37) and (38) are nonintegrable, we calculate these trajectories numerically by means of a shooting technique. The corresponding solutions are shown in Figs. 6(a) and 6(b) for two kinds of the separatrix solutions: corresponding to the first and second quadrant [i.e., $\phi_1, \phi_2 > 0$, for Fig. 6(a), and $\phi_1 < 0$, $\phi_2 > 0$, for Fig. 6(b)]. The composite dark solitons consist of two symmetric (antisymmetric) semi-infinite kinks belonging to the two counterpropagating modes of lattice vibrations. These solutions are remarkably similar to the self-induced gap solitons of the wavelength-four modes found analytically in Sec. IV and shown in Figs. 3(a) and 3(b). As a matter of fact, both these solutions are particular cases of more general kink-type solutions of the coupled equations (14) and (15) which include both the first- and second-order derivatives. This simply means that we may also treat the solutions presented in Figs. 6(a) and 6(b) as self-induced gap solitons when one of the counterpropagating modes creates an effective periodic potential to the other mode and then it localizes the counterpropagating oscillations as has been explained for the case of the wavelength-four

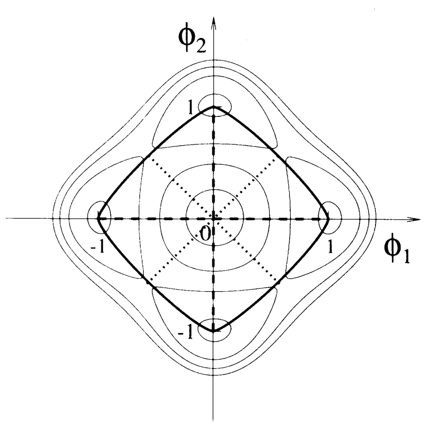

FIG. 5. Contour plot of the potential $V(\phi_1, \phi_2)$ and three types of the separatrix curves (shown by solid, dashed, and dotted lines) connecting adjacent extrema (maxima or saddle) points.

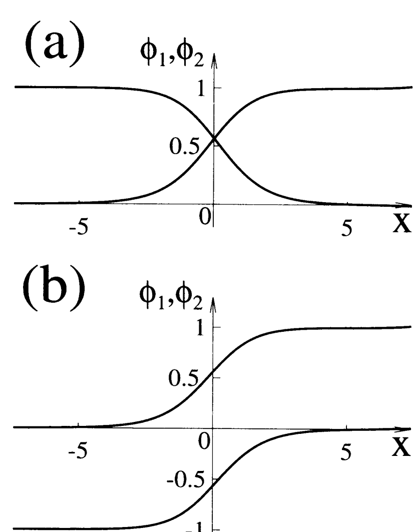

FIG. 6. Solitary-wave envelopes corresponding to two separatrices shown by solid lines in Fig. 5: (a) $\phi_1, \phi_2 > 0$, (b) $\phi_1 < 0$, $\phi_2 > 0$. Note the similarity between these solutions and those presented in Figs. 3(a) and 3(b).

modes. In the case considered in Sec. IV, however, the physical interpretation is more straightforward because these two modes in the lattice may be connected in a simple manner with vibrations of odd and even particle. This shows directly *how* the effective periodic potential is created by the second counterprogating mode: it is produced by vibrations of the other (odd or even) group of particles in the lattice.

VI. PARAMETRICALLY DRIVEN DAMPED LATTICES

The nonlinear standing modes analyzed above correspond to the case of undriven and undamped oscillations. However, a realistic physical model includes damping which may be compensated by applying a (direct or parametric) external force. Turning now to possible physical realizations of the standing nonlinear modes, we refer to the recent experiments with the driven damped chain of nonlinear pendulums [12] which may be described by the equation

$$m\frac{d^2u_n}{dt^2} - k_2(u_{n+1} + u_{n-1} - 2u_n) + m\omega_0^2 \sin u_n = F\,\cos(2\omega_e t)\sin u_n - \gamma\frac{du_n}{dt}. \quad (46)$$

In the absence of the right-hand side, Eq. (46) is a particular form of Eq. (1) at $\alpha = 0$ and $\beta = m\omega_0^2/6$ provided we expand $\sin(u_n)$ in the Taylor series, $\sin(u_n) \approx u_n - (1/6)u_n^3$.

If the force ($\sim F$) and damping ($\sim \gamma$) are small, we may try to find the condition for *stabilization* of the damping of the nonlinear modes by the parametric driving force. In the case of breather solitons (the simplest spatially localized modes) such a problem was first considered in Ref. [30] (see also more recent studies in Refs. [31,32] where chaotic regimes were described as well). To analyze nonlinear localized modes in the driven damped chain, we look for forced solutions of Eq. (46) in the form,

$$u_n(t) = f_n \cos(\omega_e t + \Theta), \quad (47)$$

and keep only two lowest-order terms in the Taylor series. Using the so-called rotating-wave approximation we take into account only the first harmonic in Eq. (46) and obtain the equation for the real function f_n which describes the wave envelope,

$$[m(\omega_0^2 - \omega_e^2) - (F/2)\cos(2\phi)]f_n = k_2(f_{n+1} + f_{n-1} - 2f_n) + \frac{3}{4}\beta f_n^3, \quad (48)$$

where the phase shift Θ is given by the relation

$$\sin(2\Theta) = \frac{2\gamma\omega_e}{F}. \quad (49)$$

Equation (48) is useful to investigate *forced oscillations* of localized modes on a standing carrier wave with the wave number q in the lattice. The cases $q = 0$ (parametric stabilization of a breather) and $q = \pi/a$ (parametric stabilization of a cutoff kink) have been analyzed earlier in Refs. [30] and [13], respectively. As a matter of fact, a general case of an arbitrary q can be reduced to the system (37) and (38) under special assumptions. Here, however, we will briefly analyze the case of $q = \pi/2a$ which is characterized by the exact solutions in the form (31) and (32).

Following Sec. IV, we reduce Eq. (48) for the envelopes of the two counterpropagating waves Ψ_1 and Ψ_2 to a system of two equations for the functions w and v [see Eq. (19)] describing, respectively, the amplitudes of odd and even particle oscillations. Assuming that the functions w and v are time independent (certainly realized for steady-state forced oscillations), we finally come to the system (27), (28) where this time the parameters κ and λ are determined by the relations

$$\kappa = \left[m(\omega_0^2 - \omega_e^2) - 2k_2 - \frac{1}{2}F\cos(2\Theta)\right], \quad (50)$$

$$\lambda = (3/4)\beta. \quad (51)$$

Thus, we come naturally to the conclusion that standing localized modes in this case are described by the same solutions (31) and (32) with the parameters fixed by the amplitude and frequency of the parametric driving force. In particular, the steady-state amplitude f_0 of the kink solitons is given by the formula

$$f_0^2 = \frac{4}{3\beta}\left[m(\omega_0^2 - \omega_e^2) - 2k_2 - \frac{1}{2}F\cos(2\Theta)\right], \quad (52)$$

where the phase shift Θ has to be found from Eq. (49). As a matter of fact, this type of localized modes was observed experimentally in a damped and parametrically driven chain of pendulums [12].

The results displayed above are, in fact, based on the approximation of the NLS equation. However, numerical analysis of the simplest case of the spatially localized (breather) modes shows that such an approximation is valid, however, for rather small amplitudes of the driving force F only. For very large values of the amplitude F the forced oscillations may become much more complicated, displaying period doubling sequences and chaos [32]. This kind of scenario may be qualitatively expected for the other types of localized solutions, in particular those analyzed here, and more detailed analytical and numerical studies will be presented elsewhere.

VII. CONCLUSIONS

In conclusion, taking the well-known model of a chain with nonlinear (cubic and quartic) on-site potential and using the approximation based on the discrete nonlinear Schrödinger equation, we have developed the theory of standing localized modes in discrete lattices. We have shown that this kind of localized mode is described by a system of two coupled NLS equations with nonlinear coupling terms of three different kinds, and this system seems to cover all the particular cases known up to now. It displays also types of localized solutions that have no simple analog in the theory of solitons on a traveling carrier wave. In particular, our model includes, as a particular case, the so-called self-induced gap solitons, and it displays also a richer class of the similar solutions describing domain walls of lattice vibrations.

In the present paper, we have used the approximation of slowly varying envelopes which reduces to partial differential equations for the envelopes of two counterpropagating waves. When the discreteness effects become important (this is valid either for a weak coupling in the lattice or for larger nonlinearities), the wave envelopes cannot be described by equations of the continuous approximation, and we expect to observe a different kind of physical phenomena. One of the discreteness-induced effects is the existence of the so-called Peierls-Nabarro effective periodic potential which simply means that the energy of a localized mode depends on its position in the lattice. In particular, recently it was shown analytically [33] and numerically [8] that the spatially localized mode centered at a particle site is stable, whereas the mode centered between the neighboring sites is unstable. We also expect to observe this kind of effect for all the modes described here in the approximation of continuous envelopes, including the case of dark-soliton modes and self-induced gap solitons.

Finally, we would like to note that the model we presented here and the properties of its localized solutions are quite general and are to be expected in other types of discrete nonlinear lattices. For example, recently it was proven theoretically and experimentally that nonlinear localized traveling waves may propagate in an experimental electrical transmission line made of N ($N = 45$) nonlinear electric cells [34]. As a matter of fact, in the case when the intensity of a reflected wave is not suppressed (i.e., just the opposite to the case considered in [34]), we may naturally expect a strong interaction of two counterpropagating waves and creation of localized modes similar to those described in the present paper.

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